Topology and its Applications 157 (2010) 669–673

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Ultra-$\mathfrak{m}$-separability

Piotr Niemiec

Jagiellonian University, Institute of Mathematics, ul. Łojasiewicza 6, 30-348 Kraków, Poland

ARTICLE INFO

Article history:
Received 26 January 2009
Received in revised form 13 November 2009

MSC:
54E35
54A25
54C35

Keywords:
Katětov maps
Collinearity property

ABSTRACT

A metric space X is ultra-$\mathfrak{m}$-separable if the weight of the Katětov hull, $E(X)$, of X is no greater than $\mathfrak{m}$. It is shown that the collection of all nonempty ultra-$\mathfrak{m}$-separable subsets of X is an ideal closed under taking the limit of its members with respect to the Hausdorff distance. As an application of this, it is proved that if (K, d_K) is precompact and (X, d_X) is ultra-$\mathfrak{m}$-separable, then $(K \times X, D)$ is ultra-$\mathfrak{m}$-separable as well, where D is any metric on $K \times X$ such that $D((u,x),(u,y)) = d_X(x,y)$ and $D((u,x),(v,x)) = d_K(u,v)$ for any $u, v \in K$ and $x, y \in X$. Bounded ultra-$\mathfrak{m}$-separable spaces are characterized by means of their metrically discrete subsets.

Katětov maps play an important role in studying Urysohn universal spaces and are important tools in extending isometries. It turned out to be important to characterize metric spaces X for which the space $E(X)$ of all Katětov maps on X is separable. This was done by Melleray [4]. These spaces have the so-called *collinearity property*, which was investigated also by Kalton [2]. Combined results of Melleray and Kalton show that if X and Y are two metric spaces such that $E(X)$ and $E(Y)$ are separable, then $E(X \times Y)$ is also separable, when $X \times Y$ is considered with the 'maximum' or the 'sum' metric induced by the metrics of X and Y. In the opposite, $E(\mathbb{R})$ (with respect to the natural metric on $\mathbb{R}$) is separable, while $E(\mathbb{R}^2)$ is not, when $\mathbb{R}^2$ is equipped with the standard Euclidean metric. In this paper we shall show that if X is precompact and $E(Y)$ is separable, then $E(X \times Y)$ is separable with respect to any metric D on $X \times Y$ such that $D((a,u),(a,v)) = d_Y(u,v)$ and $D((a,u),(b,u)) = d_X(a,b)$ for all $a, b \in X$ and $u, v \in Y$. Since the presented proofs can easily be adapt to nonseparable spaces, all results are formulated and proved in general settings.

In this paper $\mathbb{R}_+$ stands for the set of all nonnegative reals. The Hausdorff distance induced by a metric d is denoted by dist_d. Note that dist_d takes values in $[0, +\infty]$. The weight of a topological space X is denoted by $w(X)$. We call a metric space (A, d) *metrically discrete* if there is $\varepsilon > 0$ such that $d(x, y) \geqslant \varepsilon$ for any two distinct points x and y of A.

We begin with

1. Definition. A *Katětov map* on a metric space (X, d) is any function $f : X \to \mathbb{R}_+$ such that $|f(x) - f(y)| \leqslant d(x,y) \leqslant f(x) + f(y)$ for each $x, y \in X$. The *Katětov hull* of X is the space $E(X)$ of all Katětov maps on X, equipped with the metric induced by the supremum norm, denoted by $\|\cdot\|$ (Katětov maps on X may be unbounded, their difference however is always bounded). Additionally, for each $r \in [0, +\infty]$, let $E_r(X)$ be the collection of all maps $f \in E(X)$ with $f(X) \subset [0, r]$. For more on Katětov maps the reader can see [3,5,1].

The space X is *ultra-$\mathfrak{m}$-separable* (where $\mathfrak{m} \geqslant \aleph_0$) [*ultraseparable*] if $w(E(X)) \leqslant \mathfrak{m}$ [if $E(X)$ is separable]. The collection of all ultra-$\mathfrak{m}$-separable subsets of X is denoted by $\mathcal{U}_{\mathfrak{m}}\mathcal{S}(X)$ and $\mathcal{US}(X) = \mathcal{U}_{\aleph_0}\mathcal{S}(X)$. Additionally, $\mathcal{U}^*_{\mathfrak{m}}\mathcal{S}(X)$ and $\mathcal{U}^*\mathcal{S}(X)$ are the families of the appropriate nonempty subsets of X.

E-mail address: piotr.niemiec@uj.edu.pl.

0166-8641/$ – see front matter
doi:10.1016/j.topol.2009.11.009

Note that ultra-$\mathfrak{m}$-separability is not a topological invariant (even in case of comparable metrics on the same space), that is, it does depend on the metric on a metrizable space.

Ultraseparable metric spaces are characterized as follows:

2. Theorem. *A metric space (X, d) is ultraseparable if and only if X has the collinearity property, i.e. if there is no infinite subset A of X for which $\inf\{d(x, y) + d(y, z) - d(x, z)\colon x, y, z$ are distinct points of $A\} > 0$. Closed balls in the completion of an ultraseparable space are compact.*

The above result is due to Melleray [4]. Theorem 2 and the results of Kalton [2] concerning the collinearity property imply in particular that:

3. Theorem.

(i) *Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$ such that there is a finite system of points $\{(a_1^{(s)}, \dots, a_n^{(s)})\}_{s=1}^p$ of $(0, +\infty)^n$ for which $\|(x_1, \dots, x_n)\| = \max_s(\sum_{j=1}^n a_j^{(s)}|x_j|)$. If $(X_1, d_1), \dots, (X_n, d_n)$ are ultraseparable, then the space (X, d) is ultraseparable as well, where $X = X_1 \times \dots \times X_n$ and $d((x_1, \dots, x_n), (y_1, \dots, y_n)) = \|(d_1(x_1, y_1), \dots, d_n(x_n, y_n))\|$.*

(ii) *If $\|\cdot\|$ is a norm on $\mathbb{R}^n$ which is polyhedral (i.e. the closed unit $\|\cdot\|$-ball is a polyhedron), then the space $(\mathbb{R}^n, \|\cdot\|)$ is ultraseparable.*

In contrast, Kalton (in the previously cited paper) has also shown that if a norm $\|\cdot\|$ is not polyhedral, then $(\mathbb{R}^n, \|\cdot\|)$ does not have the collinearity property and therefore is not ultraseparable (while $\mathbb{R}$ clearly is). In particular, $\mathbb{R}^2$ with the Euclidean metric is not ultraseparable. This shows that the Cartesian product of two ultraseparable spaces X and Y need not be ultraseparable (for specific metrics on $X \times Y$). In the sequel we shall prove that if additionally one of the spaces X or Y is *precompact* (i.e. is totally bounded or, equivalently, its completion is compact), then $X \times Y$ is ultraseparable with respect to any metric which 'preserves' metrics of X and Y (see Corollary 8).

If A is a nonempty subset of a metric space (X, d) and $f : A \to \mathbb{R}_+$ is a Katětov map, then $\hat{f} : X \to \mathbb{R}_+$ is the *Katětov extension* of f, given by $\hat{f}(x) = \inf_{a \in A}(f(a) + d(x, a))$. Katětov's theorem [3] states that the map $E(A) \ni f \mapsto \hat{f} \in E(X)$ is isometric.

From now on, $\mathfrak{m}$ is a fixed infinite cardinal.

4. Proposition. *Let (X, d) be a metric space. If $C, D \in \mathcal{U}_\mathfrak{m}\mathcal{S}(X)$, then $\overline{C \cup D} \in \mathcal{U}_\mathfrak{m}\mathcal{S}(X)$. More generally, if $\{A_t\}_{t\in T}$ $(T \neq \emptyset)$ is a family of ultra-$\mathfrak{m}$-separable subsets of X, then the set $B = \overline{\bigcup_{t\in T} A_t}$ is ultra-$\mathfrak{n}$-separable, where $\mathfrak{n} = \mathfrak{m}^{\operatorname{card} T}$.*

Proof. It is enough to show that $A = \bigcup_{t\in T} A_t$ is ultra-$\mathfrak{n}$-separable. We may assume that each A_t is nonempty. Fix $a \in A$ and put $e_a : X \ni x \mapsto d(x, a) \in \mathbb{R}_+$. Let $\mathcal{R} = \{(f_t)_{t\in T} \in \prod_{t\in T} E(A_t)\colon \sup_{t\in T} \|f_t - e_a|_{A_t}\| < +\infty\}$ be the space equipped with the metric $p((f_t)_t, (g_t)_t) = \sup_{t\in T} \|f_t - g_t\|$. Then $w(\mathcal{R}) \leqslant \mathfrak{n}$. Indeed, if for each n and $t \in T$, $D_{n,t}$ is a dense subset of the ball $B(e_a|_{A_t}, n) \subset E(A_t)$ such that $\operatorname{card} D_{n,t} \leqslant \mathfrak{m}$, then the set $\mathcal{D} = \bigcup_{n\geqslant 1}(\prod_{t\in T} D_{n,t})$ is a dense subset of $\mathcal{R}$ with $\operatorname{card}\mathcal{D} \leqslant \mathfrak{n}$. Now put $\mathcal{R}_0 = \{(f|_{A_t})_{t\in T} \in \prod_{t\in T} E(A_t)\colon f \in E(A)\}$. The set $\mathcal{R}_0$ is contained in $\mathcal{R}$, because the map $f - e_a|_A$ is bounded for any $f \in E(A)$. Thus $w(\mathcal{R}_0) \leqslant \mathfrak{n}$. Finally, the function $\mathcal{R}_0 \ni (f_t)_{t\in T} \mapsto \bigcup_{t\in T} f_t \in E(A)$ is a well defined continuous surjection and therefore $w(E(A)) \leqslant \mathfrak{n}$ as well. □

The above result states that the family $\mathcal{U}_\mathfrak{m}\mathcal{S}(X)$ is an ideal of subsets of X which is closed under taking closures.

5. Lemma. *Let U and V be two nonempty subsets of a metric space (X, d) and let $g \in E(X)$. Then $\|\widehat{(g|_U)} - \widehat{(g|_V)}\| \leqslant 2\operatorname{dist}_d(U, V)$.*

Proof. Let $x \in X$ and $\varepsilon > 0$. It is enough to show that $\widehat{(g|_U)}(x) - \widehat{(g|_V)}(x) \leqslant 2\operatorname{dist}_d(U, V) + \varepsilon$. Take $v \in V$ such that $\widehat{(g|_V)}(x) \geqslant g(v) + d(x, v) - \varepsilon$. Then we have: $\widehat{(g|_U)}(x) - \widehat{(g|_V)}(x) \leqslant \inf_{u\in U}(g(u) + d(x, u) - g(v) - d(x, v)) + \varepsilon \leqslant 2\inf_{u\in U} d(u, v) + \varepsilon \leqslant 2\operatorname{dist}_d(U, V) + \varepsilon$. □

Now we shall prove the main result of the paper.

6. Theorem. *Let (X, d) be a metric space. If $A_n \in \mathcal{U}^*_\mathfrak{m}\mathcal{S}(X)$ $(n \geqslant 1)$ and*

$$\operatorname{dist}_d(A_n, A) \to 0 \quad (n \to +\infty) \tag{1}$$

for some nonempty subset A of X, then $A \in \mathcal{U}_\mathfrak{m}\mathcal{S}(X)$.

Proof. Let C be the closure in X of $\bigcup_{n=1}^\infty A_n$. By (1), $A \subset C$. Let $\mathcal{T}_0$ be the subset of $E(C)$ consisting of all Katětov maps f on C for which there are $n \geqslant 1$ and $g \in E(A_n)$ such that $\hat{g}|_C = f$, and let $\mathcal{T}$ be the closure in $E(C)$ of $\mathcal{T}_0$. It is enough to prove that $w(\mathcal{T}_0) \leqslant \mathfrak{m}$ and $\mathcal{T}|_A = E(A)$.

Since $w(E(A_n)) \leqslant \mathfrak{m}$, hence there is a dense subset $\mathcal{B}_n$ of $E(A_n)$ of cardinality no greater than $\mathfrak{m}$. Let $\mathcal{D}_n = \{\hat{g}|_C\colon\ g \in \mathcal{B}_n\}$ and $\mathcal{D} = \bigcup_{n=1}^{\infty} \mathcal{D}_n \subset \mathcal{T}_0$. Clearly, $\operatorname{card}\mathcal{D} \leqslant \mathfrak{m}$. We claim that $\mathcal{D}$ is dense in $\mathcal{T}_0$. Indeed, if $f \in \mathcal{T}_0$ and $\varepsilon > 0$, then there is $n \geqslant 1$ and $g \in E(A_n)$ such that $f = \hat{g}|_C$. Since $\mathcal{B}_n$ is dense in $E(A_n)$, hence there is $h \in \mathcal{B}_n$ for which $\|h - g\| \leqslant \varepsilon$. But then $\hat{h}|_C \in \mathcal{D}$ and $\|\hat{h}|_C - f\| \leqslant \|\hat{h} - \hat{g}\| = \|h - g\| \leqslant \varepsilon$.

Now let $f \in E(A)$. Put $F_n = \widehat{f_n}|_C \in \mathcal{T}_0$, where $f_n = \hat{f}|_{A_n}$. By Lemma 5, $\|F_n - F_m\| \leqslant 2\operatorname{dist}_d(A_n, A_m)$ and thus $(F_n)_n$ is a fundamental sequence. Since the Katětov hull is complete, hence there is $F \in \mathcal{T}$ such that $\lim_{n\to\infty} \|F_n - F\| = 0$. It remains to check that $F|_A = f$. Fix $a \in A$. We have: $F_n(a) - f(a) = \inf_{x\in A_n}(f_n(x) + d(x,a) - f(a)) = \inf_{x\in A_n}[\inf_{b\in A}(f(b) + d(b,x)) + d(x,a) - f(a)] = \inf_{\substack{b\in A\\ x\in A_n}} (f(b) - f(a) + d(b,x) + d(x,a))$. So,

$$F_n(a) - f(a) \leqslant 2 \inf_{x\in A_n} d(a,x) = 2\operatorname{dist}_d(a, A_n) \leqslant 2\operatorname{dist}_d(A_n, A) \tag{2}$$

and $f(b) - f(a) + d(b,x) + d(x,a) \geqslant -d(a,b) + d(b,x) + d(x,a) \geqslant 0$, which implies that $F_n(a) - f(a) \geqslant 0$. This, combined with (2), yields that $|F_n(a) - f(a)| \leqslant 2\operatorname{dist}_d(A_n, A) \to 0$ and therefore $F(a) = \lim_{n\to\infty} F_n(a) = f(a)$. □

For a metric space X, let

$$\mathfrak{m}^{[X]} = \sup\{\mathfrak{m}^{\kappa}\colon\ X \text{ contains a metrically discrete subset of power } \kappa\}.$$

The first consequence of the above result is the following

7. Theorem. *Let (X,d) and (Z,ϱ) be two nonempty metric spaces. Let $F : X \to \mathcal{U}^*_{\mathfrak{m}}\mathcal{S}(Z)$ be a uniformly continuous multifunction, i.e. for each $\varepsilon > 0$ there is $\delta > 0$ such that $\operatorname{dist}_\varrho(F(x), F(y)) \leqslant \varepsilon$, whenever $d(x,y) \leqslant \delta$. Then the set $T = \bigcup_{x\in X} F(x) \subset Z$ is ultra-$\mathfrak{n}$-separable with $\mathfrak{n} = \mathfrak{m}^{[X]}$. In particular, if X is precompact, then T is ultra-$\mathfrak{m}$-separable.*

Proof. Take a sequence $(A_n)_{n=1}^{\infty}$ of metrically discrete subsets of X such that $X = \bigcup_{x\in A_n} \bar{B}(x, \frac{1}{n})$ for each $n \geqslant 1$. Let $T_n = \bigcup_{x\in A_n} F(x)$. By Proposition 4, $w(E(T_n)) \leqslant \mathfrak{m}^{\operatorname{card} A_n}$ $(\leqslant \mathfrak{n})$ and thus T_n is ultra-$\mathfrak{n}$-separable. Thanks to Theorem 6, it is enough to show that

$$\operatorname{dist}_\varrho(T_n, T) \to 0 \quad (n \to +\infty).$$

Let $\varepsilon > 0$. Since F is uniformly continuous, hence there is $m \geqslant 1$ such that $\operatorname{dist}_\varrho(F(x), F(y)) \leqslant \varepsilon$ whenever $d(x,y) \leqslant \frac{1}{m}$. Let $n \geqslant m$ and $t \in T$. Take $x \in X$ for which $t \in F(x)$. There is $y \in A_n$ such that $d(x,y) \leqslant \frac{1}{n}$ and therefore $\operatorname{dist}_\varrho(F(x), F(y)) \leqslant \varepsilon$. This yields that $\operatorname{dist}_\varrho(t, F(y)) \leqslant \varepsilon$. So, since $F(y) \subset T_n$, thus $\operatorname{dist}_\varrho(t, T_n) \leqslant \varepsilon$. But this, combined with the connection $T_n \subset T$, gives $\operatorname{dist}_\varrho(T, T_n) \leqslant \varepsilon$ for each $n \geqslant m$, which finishes the proof. □

If d and ϱ are metrics on sets X and Y, respectively, then we say that a metric D on $X \times Y$ is *d-ϱ-preserving* if $D((x,u),(x,v)) = \varrho(u,v)$ and $D((x,u),(y,u)) = d(x,y)$ for any $x, y \in X$ and $u, v \in Y$. It is not required that a d-ϱ-preserving metric on $X \times Y$ induces the product topology.

8. Corollary. *Let (X,d) and (Y,ϱ) be two (nonempty) metric spaces and let D be a d-ϱ-preserving metric on $X \times Y$. If (Y,ϱ) is ultra-$\mathfrak{m}$-separable, then $(X \times Y, D)$ is ultra-$\mathfrak{n}$-separable with $\mathfrak{n} = \mathfrak{m}^{[X]}$. In particular, if (X,d) is precompact, then $(X \times Y, D)$ is ultra-$\mathfrak{m}$-separable.*

Proof. Let $F : X \to \mathcal{P}(X \times Y)$ ($\mathcal{P}(X \times Y)$ is the power set of $X \times Y$) be a multifunction defined by $F(x) = \{x\} \times Y$. Since D is d-ϱ-preserving, therefore $F(x)$ is isometric to Y for each $x \in X$ and $\operatorname{dist}_D(F(a), F(b)) \leqslant d(a,b)$ for every $a, b \in X$. So, $F : X \to \mathcal{U}^*_{\mathfrak{m}}\mathcal{S}(X \times Y)$ is uniformly continuous. Now Theorem 7 finishes the proof. □

Now we shall give examples dealing with Theorem 7. Let us agree that if F is a multifunction from a metric space (X,d) to the power set of a metric space (Y,ϱ), then (X,d) and (Y,ϱ) are called the *domain* and the *underlying codomain* of F, respectively.

9. Examples. In the three examples stated below F is a multifunction given by the formula $F(x) = \{x\}$. The domain and the underlying codomain of F shall be described. Note that whatever they are, $F(x)$ is always ultraseparable.

(A) Let (X,d) be a separable metric space which is not ultraseparable (e.g. $X = \mathbb{N}$ and $d =$ the discrete metric on $\mathbb{N}$). There is a metric ϱ on X, compatible with the topology of X, such that (X,ϱ) is precompact. Let (X,ϱ) and (X,d) be the domain and the underlying codomain of F, respectively. The multifunction F is *continuous*, i.e. for each $x \in X$ and $\varepsilon > 0$ there is $\delta > 0$ such that $\operatorname{dist}_d(F(x), F(y)) \leqslant \varepsilon$ whenever $\varrho(x,y) \leqslant \delta$. However, the set $\bigcup_{x\in X} F(x) = X$ is not ultraseparable (with respect to the metric d). The example explains that the assumption of Theorem 7 that F is uniformly continuous is essential.

(B) Now let d and ϱ be the natural metric and the discrete one on $\mathbb{N}$, respectively, and let $(\mathbb{N}, d)$ and $(\mathbb{N}, \varrho)$ be the domain and the underlying codomain of F. In that case F is uniformly continuous, $(\mathbb{N}, d)$ is ultraseparable, while $(\mathbb{N}, \varrho)$ is not. The example shows that if the domain X of a multifunction F is not precompact, then the union $\bigcup_{x \in X} F(x)$ may not be ultra-$\mathfrak{m}$-separable even if X and each $F(x)$ are.

(C) Let (A, d) be a discrete metric space of cardinality $\mathfrak{m} \geqslant \aleph_0$ and let (A, d) be both the domain and the underlying codomain of F. Again, F is clearly uniformly continuous and, by Theorem 7, (A, d) is ultra-$(\aleph_0)^{\mathfrak{m}}$-separable. But $(\aleph_0)^{\mathfrak{m}} = 2^{\mathfrak{m}}$ and $w(E(A)) = 2^{\mathfrak{m}}$. This example shows that the claim of Theorem 7 cannot be improved in general.

Theorem 6 implies the following

10. Proposition. *If (X, d) is an infinite metric space and $(A_n)_{n=1}^{\infty}$ is a sequence of metrically discrete subsets of X such that*

$$X = \bigcup_{a \in A_n} \bar{B}\left(a, \frac{1}{n}\right)$$

for each n, then $w(E(X)) = \sup_n w(E(A_n)) = \sum_{n=1}^{\infty} w(E(A_n))$. In particular, $w(E(X)) \leqslant 2^{[X]} \leqslant 2^{w(X)}$.

The foregoing result shows that it is enough to study the weights of the Katětov hulls of metrically discrete ones.

In case of a bounded metric space X the weight of $E(X)$ can be easily computed. Namely,

11. Proposition. *If (X, d) is a* ***bounded*** *infinite metric space and $r \in (\frac{1}{2} \operatorname{diam} X, +\infty]$, then $w(E_r(X)) = 2^{[X]}$. In particular, $E_r(X)$ and $E(X)$ have the same weight.*

Proof. By Proposition 10, it suffices to show that $w(E_r(X)) \geqslant 2^{\operatorname{card} A}$ for each metrically discrete subset A of X. Take such a number $\delta \in (0, r - \frac{1}{2} \operatorname{diam} X)$ that $d(x, y) \geqslant \delta$ for every two distinct points x and y of A. Observe that if $f : A \to \{r, r - \delta\}$, then f is Katětov and so is $f^* = \min(\hat{f}, r)$. Moreover, f^* coincides with f on A. It is easy to see that the map $\{r, r - \delta\}^A \ni f \mapsto f^* \in E_r(X)$ is isometric. So, the notice that the set $\{r, r - \delta\}^A$ is metrically discrete and of cardinality $2^{\operatorname{card} A}$ finishes the proof. $\square$

Proposition 11 gives a simple formula for the weight of the Katětov hull of a bounded metric space. The unbounded case is more complicated in general and we shall only prove the next few results on them. To do this, we recall that an infinite cardinal $\mathfrak{m}$ has *countable cofinality* if there is a sequence $(\mathfrak{m}_n)_n$ of cardinals less than $\mathfrak{m}$ such that $\mathfrak{m} = \sup_n \mathfrak{m}_n$ (otherwise $\mathfrak{m}$ has uncountable cofinality).

12. Proposition. *Let X be an unbounded metric space. If X contains no metrically discrete subset of cardinality $w(X)$ or if X contains a bounded metrically discrete subset of cardinality $w(X)$, then $w(E(X)) = 2^{[X]}$.*

Proof. Note that for each $\mathfrak{n} < \mathfrak{m}$ there is a bounded metrically discrete subset A of X such that $\operatorname{card} A \geqslant \mathfrak{n}$ and apply Propositions 10 and 11. $\square$

Since every infinite metric space X whose weight has uncountable cofinality contains a bounded metrically discrete subset of cardinality $w(X)$, thus the above result implies the following

13. Corollary. *If the weight of an infinite metric space X has uncountable cofinality, then $w(E(X)) = 2^{w(X)}$.*

Note that every infinite cardinal λ which is not limit or which is of the form $\kappa^{\mathfrak{n}}$, where $\kappa \geqslant 2$ and $\mathfrak{n}$ is infinite, has uncountable cofinality and therefore Corollary 13 gives a simple formula for $w(E(X))$ in case of a metric space X such that $w(X) = \lambda$.

We end the paper with the following result, which can be deduced from the part of the proof of Theorem 2.8 of [5].

14. Proposition. *If (X, d) is an infinite metric space such that*

$$\inf\bigl\{d(x, y) + d(y, z) - d(x, z)\colon\ x, y, z \text{ are distinct points of } X\bigr\} > 0,$$

then X is ultra-$\mathfrak{n}$-separable with $\mathfrak{n} = 2^{\operatorname{card} X}$.

References

[1] S. Gao, A.S. Kechris, On the classification of Polish metric spaces up to isometry, Mem. Amer. Math. Soc. 161 (2003), viii+78 pp.
[2] N. Kalton, Extending Lipschitz maps into $\mathcal{C}(K)$-spaces, Israel J. Math. 162 (2007) 275–316.
[3] M. Katětov, On universal metric spaces, in: Frolik (Ed.), General Topology and its Relations to Modern Analysis and Algebra VI, Proceedings of the Sixth Prague Topological Symposium 1986, Helderman Verlag, Berlin, 1988, pp. 323–330.
[4] J. Melleray, On the geometry of Urysohn's universal metric space, Topology Appl. 154 (2007) 384–403.
[5] J. Melleray, Some geometric and dynamical properties of the Urysohn space, Topology Appl. 155 (14) (2008) 1531–1560.